Meeting 3

SOLVONG LP PROBLEMS

BY MEANS OF GRAPHICAL METHOD

THREE WAYS OF SOLVING AN LP PROBLEM

1. Graphical solution (practically only for a case of two decision variables; advantage: main issues covered in an illustrative and comprehensible manner)

2. SIMPLEX algorithm (universal method to deal with all types of LP problems; iterative procedure that enables determination of optimal solution or identification of an infeasible, unbounded or multiple optima problem)

3. Computer programme that uses either SIMPLEX or another optimization procedure (we will use Excel to deal with LP problems)

Steps for Graphical Method

Step 1 — Formulate the LPP

Step 2 — Construct a graph and plot the constraint lines

Step 3 — Determine the valid side of each constraint line

Step 4 — Identify the feasible solution region

Step 5 — Find the optimum points

Step 6 — Calculate the co-ordinates of optimum points

Step 7 — Evaluate the objective function at optimum points to get the required maximum/minimum value of the objective function

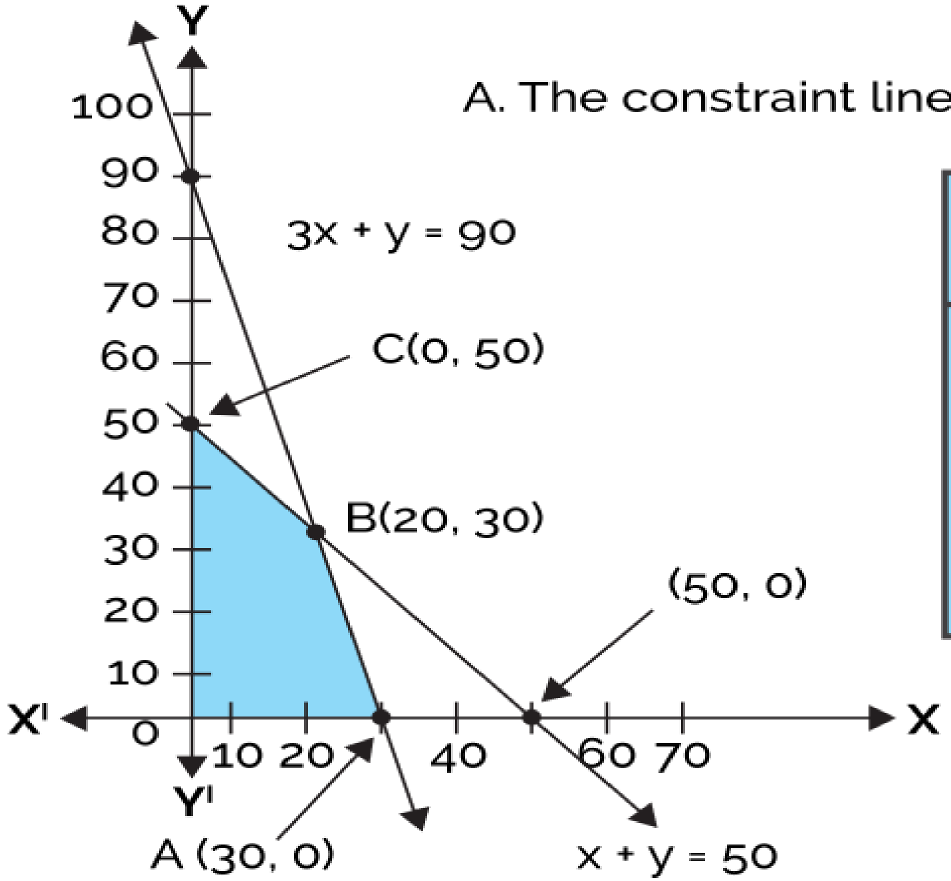
<u>Example 1</u>. (the case of unique optimal solution)

Q. Maximize and minimize $z = 4x + y$ subject to:- $x + y \le 50$

$3x + y \le 90$

$x \ge 0, y \ge 0$

A. The constraint lines are $x + y = 50$, $3x + y = 90$, $x = 0$, $y = 0$



Corner Point	Corresponding value of Z
(0, 0)	0
(30, 0)	120
(20, 30)	110
(0, 50)	50

Hence, maximum value of Z is 120 at the point (30, 0) and the minimum value of z is 0 at the point (0, 0).

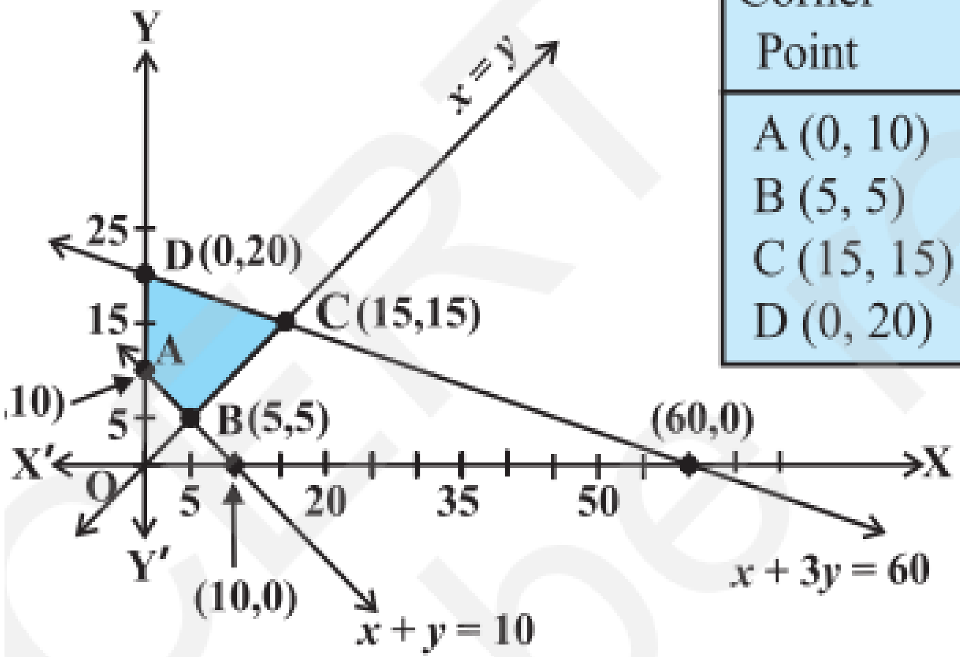
The corner point in B is computed through solving the following set of equations:

$$\begin{cases} x + y = 50 \\ 3x + y = 90 \end{cases} \Rightarrow x = 20, y = 30$$

<u>Example 2</u>. (the case of multiple solutions)

Solve graphically the following LP problem; consider both minimization and maximization of the following objective function: $f(x,y) = 3x + 9y$

$$x + 3y \le 60$$
$$x + y \ge 10$$
$$x \le y$$
$$x \ge 0,\ y \ge 0$$



Corner Point	Corresponding value of $Z = 3x + 9y$	
A (0, 10)	90	
B (5, 5)	**60** ←	Minimum
C (15, 15)	**180**	Maximum (Multiple optimal solutions)
D (0, 20)	**180**	

The corner points in B and C are computed through solving the following sets of equations:

Point B: $\begin{cases} \boldsymbol{x + y = 10} \\ \boldsymbol{x = y} \end{cases} \Longrightarrow \boldsymbol{x = 5, y = 5}$

Point C: $\begin{cases} \boldsymbol{x + 3y = 60} \\ \boldsymbol{x = y} \end{cases} \Longrightarrow \boldsymbol{x = 15, y = 15}$

If there are two corner points with the optimal solutions then all the points located in the interval linking such two points also give optimal solution, which can be computed as follows (so called linear combination):

$$(\boldsymbol{x_{opt}}, \boldsymbol{y_{opt}}) = \omega \cdot (x_{1opt}, y_{1opt}) + (1 - \omega) \cdot (x_{2opt}, y_{2opt})$$

where:

$0 \leq \omega \leq 1$ denotes a linear weight

e.g. point (x=7.5; y=17.5) = 0.5 * (point D, being: x=0, y=20) + 0.5 * (point C, being: x=15, y=15)

was computed for the weight $\omega = 0.5$

See that our objective function *f*(x=7.5; y=17.5) = 7.5*3 + 9*17.5 = **180 (which is the same value as in points B&C)**

<u>Example 3</u>. (the case of unbounded optimal solutions)

Solve by graphical method

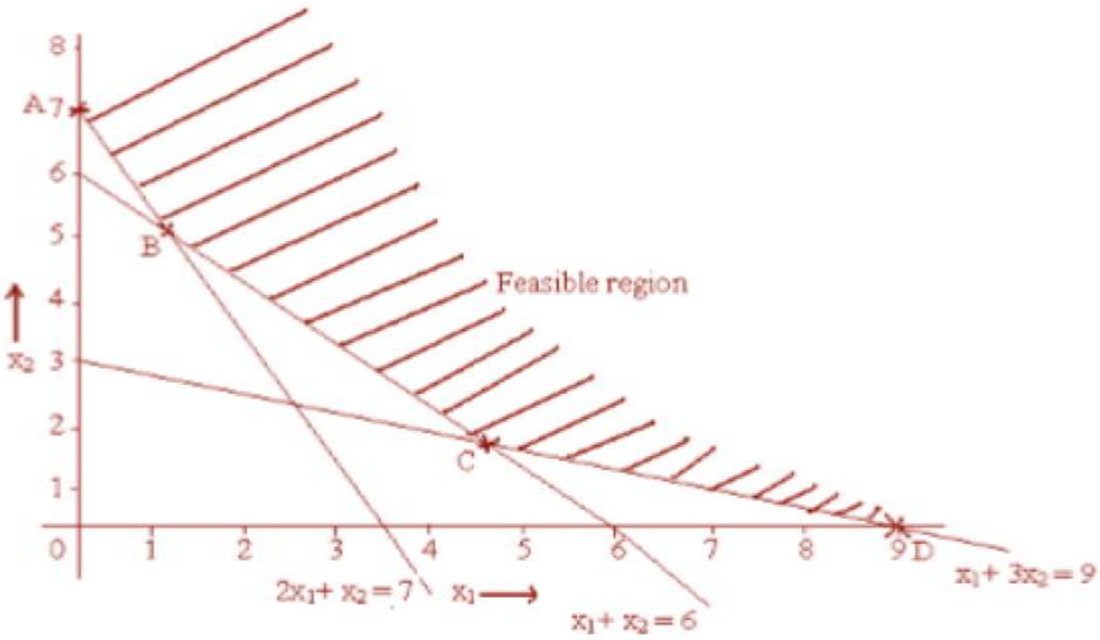
Max $Z = 3x_1 + 5x_2$

Subject to $2x_1 + x_2 \geq 7$

$x_1 + x_2 \geq 6$

$x_1 + 3x_2 \geq 9$

and $x_1 , x_2 \geq 0$



No limitations to maximization !

The corner points in B and C are computed through solving the following sets of equations:

Point B: $\begin{cases} 2x_1 + x_2 = 7 \\ x_1 + x_2 = 6 \end{cases} \Longrightarrow x_1 = 1,\ x_2 = 5$

Point C: $\begin{cases} x_1 + x_2 = 6 \\ x_1 + 3x_2 = 9 \end{cases} \Longrightarrow x_1 = 4.5,\ x_2 = 1.5$

<u>Example 4</u>. (the case of no solution)

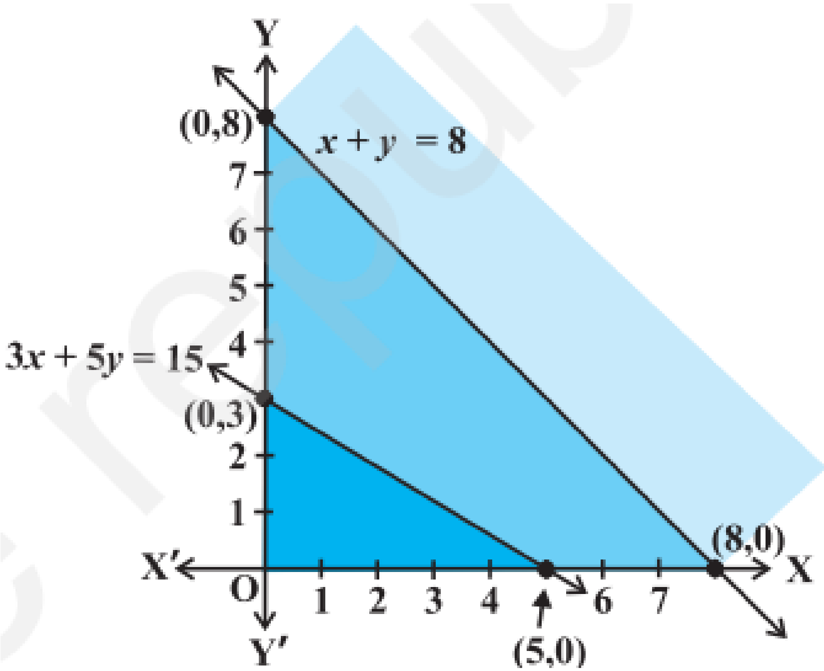
Minimize or maximize the following LP problem:

$f(x,y) = 3x + 2y$

$$x + y \geq 8$$

$$3x + 5y \leq 15$$

$$x \geq 0,\ y \geq 0$$



No feasible solution exist!

EXAMPLE 5 (graphical method)

The Healthy Pet Food Company manufactures two types of dog food: Meaties and Yummies. Each package of Meaties contains 2 pounds of cereal and 3 pounds of meat; each package of Yummies contains 3 pounds of cereal and 1.5 pounds of meat. Healthy believes it can sell as much of each dog food as it can make. Meaties sell for $2.80 per package and Yummies sell for $2.00 per package. Healthy's production is limited in several ways. First, Healthy can buy only up to 400,000 pounds of cereal each month at $0.20 per pound. It can buy only up to 300,000 pounds of meat per month at $0.50 per pound. In addition, a special piece of machinery is required to make Meaties, and this machine has a capacity of 90,000 packages per month. The variable cost of blending and packing the dog food is $0.25 per package for Meaties and $0.20 per package for Yummies. This information is given in Table B-1.

Table B-1 Healthy Pet Food Data

	Meaties	Yummies
Sales price per package	$2.80	$2.00
Raw materials per package		
Cereal	2.0 lb.	3.0 lb.
Meat	3.0 lb.	1.5 lb.
Variable cost—blending and packing	$0.25 package	$0.20 package
Resources		
Production capacity for Meaties	90,000 packages per month	
Cereal available per month	400,000 lb.	
Meat available per month	300,000 lb.	

Suppose you are the manager of the Dog Food Division of the Healthy Pet Food Company. Your salary is based on division profit, so you try to maximize its profit. How should you operate the division to maximize its profit and your salary?

Decision variables:

M = number of packages of Meaties to make each month
Y = number of packages of Yummies to make each month

The profit per package for each dog food is computed as follows:

	Meaties	Yummies
Selling price	2.80	2.00
Minus		
Meat	1.50	0.75
Cereal	0.40	0.60
Blending	0.25	0.20
Profit per package	0.65	0.45

The LP model:

$$
\begin{aligned}
\text{Maximize } z &= 0.65M + 0.45Y \\
\text{Subject to } 2M + 3Y &\le 400{,}000 \\
3M + 1.5Y &\le 300{,}000 \\
M &\le 90{,}000 \\
M, Y &\ge 0
\end{aligned}
$$

THE GEOMETRY OF LINEAR PROGRAMS

▶ A **feasible solution** satisfies all of the constraints.

▶ **Feasible set**
Set containing all of the feasible solutions.

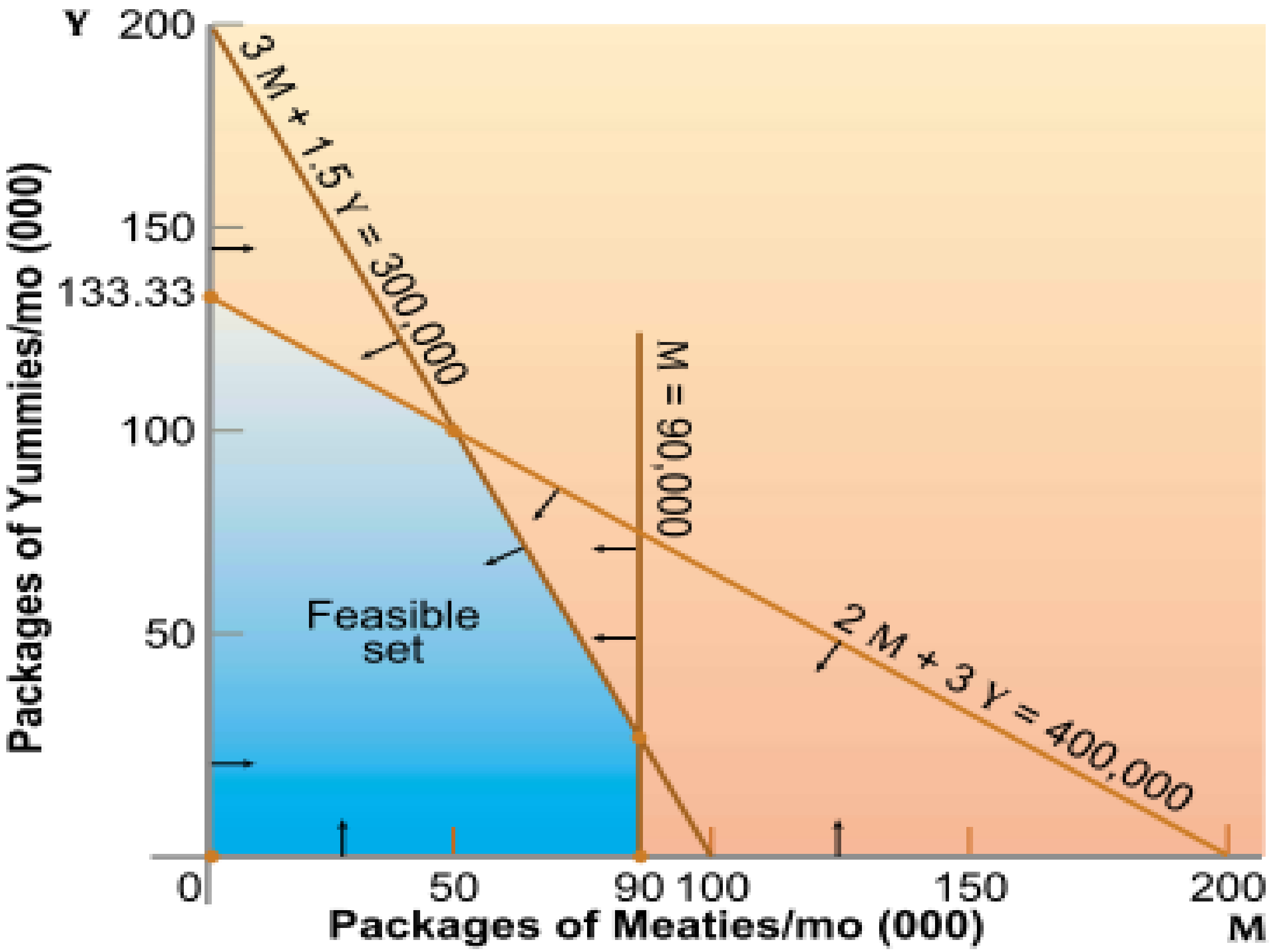
▶ **Optimal solution**
Feasible solution that produces the best objective function value.

The characteristic that makes linear programs easy to solve is their simple geometric structure. Let's define some terminology. A solution for a linear program is any set of numerical values for the variables. These values need not be the best values and do not even have to satisfy the constraints or make sense. For example, in the Healthy Pet Food problem, $M = 25$ and $Y = -800$ is a solution, but it does not satisfy the constraints, nor does it make physical sense. A **feasible solution** is a solution that satisfies all of the constraints. The **feasible set** or **feasible region** is the set of all feasible solutions. Finally, an **optimal solution** is the feasible solution that produces the best objective function value possible. Figure B-1 shows the relationships among these types of solutions.

Let's use the Healthy Pet Food example to show the geometry of linear programs and to show how two-variable problems can be solved graphically. The linear programming formulation for the Healthy Pet Food problem is:

$$\text{Maximize} \quad z = 0.65M + 0.45Y$$

$$\begin{aligned} \text{Subject to} \quad 2M + 3Y &\leq 400{,}000 \\ 3M + 1.5Y &\leq 300{,}000 \\ M &\leq 90{,}000 \\ M, Y &\geq 0 \end{aligned}$$



The corner points in the graph are computed through solving the following sets of equations:

$$\begin{cases} 3M + 1.5Y = 300000 \\ M = 90000 \end{cases} \Longrightarrow \boldsymbol{M = 90000,\ Y = 20000} \qquad \begin{cases} 3M + 1.5Y = 300000 \\ 2M + 3Y = 40000 \end{cases} \Longrightarrow \boldsymbol{M = 50000,\ Y = 100000}$$

In the end there are 5 corner points where one should search for the optimal solution of the problem under investigation. The values of the objective function, ***f*(M, Y) = 0.65*M + 0.45*Y**, in these points are as follows:

f(*M*=0, *Y*=0) = 0.65*0 + 0.45*0 = 0

f(*M*=90000, *Y*=0) = 0.65*90000 + 0.45*Y = 58500

f(*M*=90000, *Y*=20000) = 0.65*90000 + 0.45*20000 = 67500

f(*M*=50000, *Y*=100000) = 0.65*50000 + 0.45*100000 = 77500 (MAX)

f(*M*=0, *Y*=1333333) = 0.65*0 + 0.45*133333.(3) = 60000

We begin the solution process by finding the feasible set. The geometric representation of a linear equality is the set of points that lie on and to one side of the line obtained by replacing the inequality sign with an equality sign.

The constraint $M \geq 0$ restricts us to the points on or to the right of the vertical axis (the line $M = 0$). The constraint $Y \geq 0$ restricts us to the points on or above the horizontal axis. Next, we draw the constraint $2M + 3Y \leq 400{,}000$. To find the points that satisfy this inequality, we construct the line $2M + 3Y = 400{,}000$ by finding two points that lie on the line and then constructing a line through these points. The easiest points to find on the line are the ones that lie on the two axes. First, set $M = 0$ and solve for Y. This yields the point ($M = 0$, $Y = 133{,}333.22$). We then set $Y = 0$ and solve for M. This yields the point ($M = 200{,}000$ and $Y = 0$). This line is plotted on Figure B-2.

We now determine on which side of the line the points satisfy the constraint. If one point satisfies the constraint, then all points on the same side of the line satisfy the constraint. If one point does not satisfy the constraint, then no point on that side of the line satisfies the constraint, but all the points on the opposite side of the line do satisfy the constraint. It makes sense to select a simple point with which to work, such as ($M = 0$, $Y = 0$). This point satisfies the constraint $2M + 3Y \leq 400{,}000$. Therefore, all points to the lower left do also. The points to the upper right of the line represent product mixes that require more than 400,000 pounds of cereal each month and can be eliminated from consideration.

We do the same thing for the meat constraint: $3M + 1.5Y \leq 300{,}000$. We find two points on the line $3M + 1.5Y = 300{,}000$. We first set $M = 0$ and solve for Y, and then set $Y = 0$ and solve for M, yielding the points ($M = 0$, $Y = 200{,}000$) and ($M = 100{,}000$, $Y = 0$). Checking a point on one side of the line shows that the points on or to the lower left of the line are the ones that satisfy the constraint.

The final constraint, $M \leq 90{,}000$, is satisfied by the points that lie on or to the left of the vertical line $M = 90{,}000$. The feasible set is the set of points in the five-sided shaded area in Figure B-2. The feasible set for a linear program will always have a shape like the one in this problem, with edges that are straight lines and corners where the edges meet. The corners of the feasible set are called **extreme points**. Note that each extreme point is formed by the intersection of two or more constraints.

The fundamental theorem of linear programming is: If a finite optimal solution exists, then at least one extreme point is optimal. To find the exact coordinate values for the optimum from the graph:

1. We identify the constraints that intersect to form the extreme point.
2. We solve simultaneously the equations corresponding to the constraints to find the point that lies on both lines (the extreme point).

In this example, the optimal extreme point is formed by the intersection of the lines $2M + 3Y = 400{,}000$ and $3M + 1.5Y = 300{,}000$.

To solve equations simultaneously, we use the following property. We can either (1) multiply any constraint by a nonzero constant or (2) add or subtract a multiple of any equation to or from any other equation, without changing the set of solutions that the equations have simultaneously.

Therefore, to solve simultaneously the equations

$$\begin{aligned} 2M + 3Y &= 400{,}000 \\ 3M + 1.5Y &= 300{,}000 \end{aligned}$$

we can subtract two times the second equation from the first equation, leaving

$$\begin{aligned} -4M \qquad &= -200{,}000 \\ 3M + 1.5Y &= 300{,}000 \end{aligned}$$

Finally, solution is:

M = 50000

Y = 100000

Example 6.

A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

Solution Let x and y be the number of packets of food P and Q respectively. Obviously $x \geq 0$, $y \geq 0$. Mathematical formulation of the given problem is as follows:

Minimise $Z = 6x + 3y$ (vitamin A)

subject to the constraints

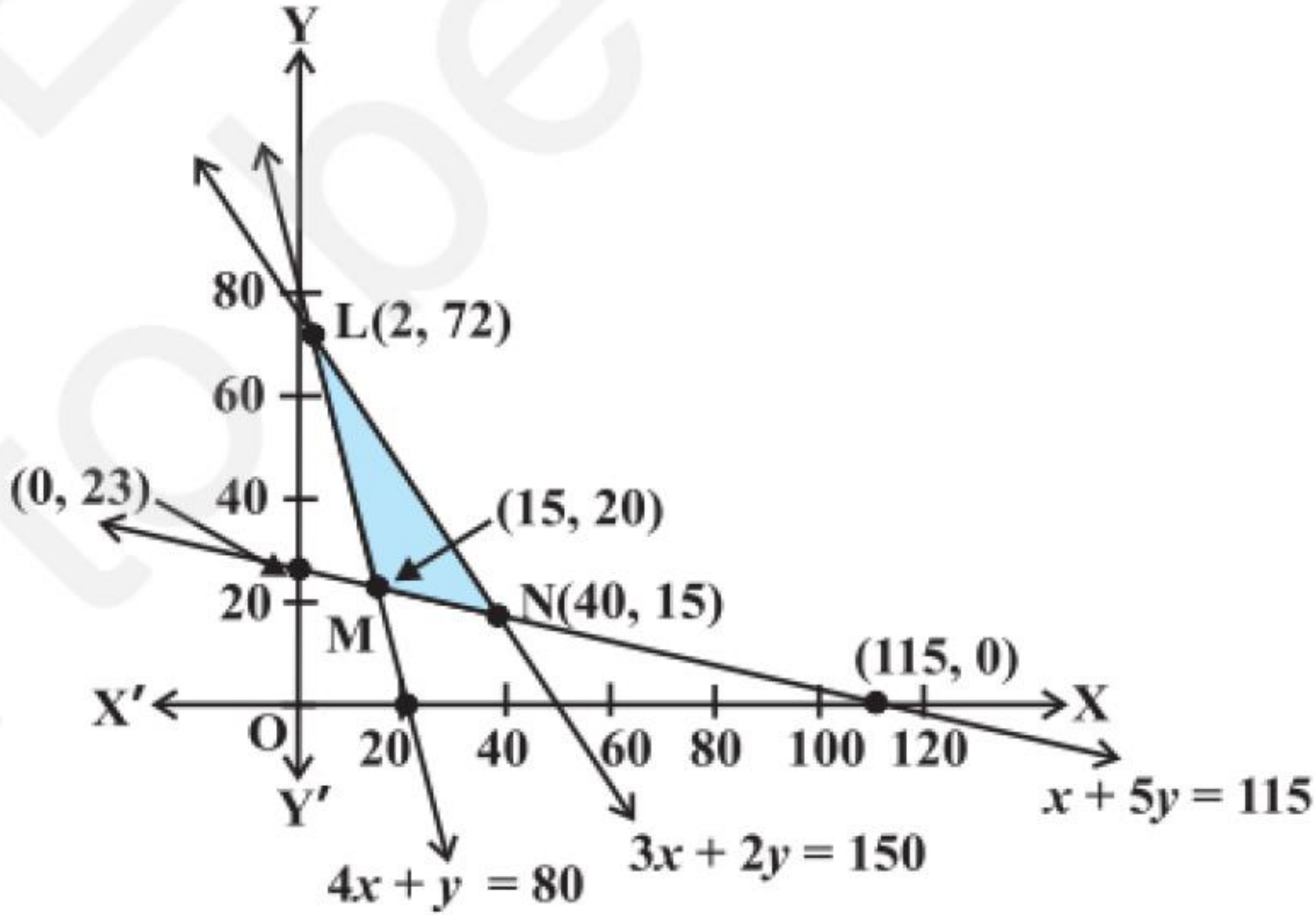
$12x + 3y \geq 240$ (constraint on calcium), i.e. $\quad 4x + y \geq 80 \quad$ ... (1)

$4x + 20y \geq 460$ (constraint on iron), i.e. $\quad x + 5y \geq 115 \quad$ ... (2)

$6x + 4y \leq 300$ (constraint on cholesterol), i.e. $\quad 3x + 2y \leq 150 \quad$ ... (3)

$x \geq 0, y \geq 0 \quad$ ... (4)

Let us graph the inequalities (1) to (4).


Y
80
L(2, 72)
60
(0, 23)
40
(15, 20)
20
N(40, 15)
M
(115, 0)
X'
X
O
20
40
60
80
100
120
Y'
x + 5y = 115
3x + 2y = 150
4x + y = 80

The coordinates of the corner points L, M and N are (2, 72), (15, 20) and (40, 15) respectively. Let us evaluate Z at these points:

Corner Point	$Z = 6x + 3y$	
(2, 72)	228	
(15, 20)	**150** $\leftarrow$	Minimum
(40, 15)	285	

From the table, we find that Z is minimum at the point (15, 20). Hence, the amount of vitamin A under the constraints given in the problem will be minimum, if 15 packets of food P and 20 packets of food Q are used in the special diet. The minimum amount of vitamin A will be 150 units.

Example 7.

A cooperative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 litres and 10 litres per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so as to maximise the total profit of the society?

Solution Let x hectare of land be allocated to crop X and y hectare to crop Y. Obviously, $x \geq 0, y \geq 0$.

Profit per hectare on crop X = Rs 10500

Profit per hectare on crop Y = Rs 9000

Therefore, total profit = Rs $(10500x + 9000y)$

The mathematical formulation of the problem is as follows:

Maximise $Z = 10500\,x + 9000\,y$

subject to the constraints:

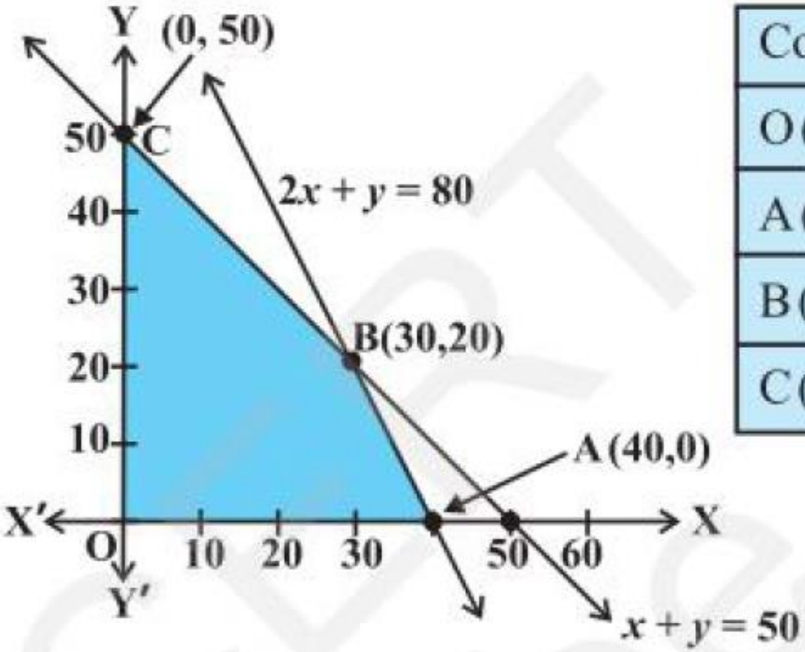
$x + y \leq 50$ (constraint related to land) ... (1)

$20x + 10y \leq 800$ (constraint related to use of herbicide)

i.e. $2x + y \leq 80$ ... (2)

$x \geq 0, y \geq 0$ (non negative constraint) ... (3)

The coordinates of the corner points O, A, B and C are (0, 0), (40, 0), (30, 20) and (0, 50) respectively. Let us evaluate the objective function Z = 10500 x + 9000y at these vertices to find which one gives the maximum profit.



Corner Point	$Z = 10500x + 9000y$	
O(0, 0)	0	
A(40, 0)	420000	
B(30, 20)	**495000** ←	Maximum
C(0,50)	450000	